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PHYSICS AND
MATHEMATICS

ON THE INVERSION OF CERTAIN MATRICES

by

Samuel Schechter
February 1, 1959

Institute of Mathematical Sciences

NEW YORK UNIVERSITY
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Abstract

Properties of matrices of the form $H = \{(a_i - b_j)^{-1}\}$ are investigated. If $G = H^{-1} = c_{ij}$ then explicit formulas are given for $\{c_{ij}\}$, and the row and column sums of G. These extend formulas previously given by Collar and Smith for the generalized Hilbert matrix. For H symmetric and positive definite the smallest eigenvalue of H and its P- condition are estimated.

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On the Inversion of Certain Matrices

1. Introduction.

Let $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$ be $2n$ distinct, but otherwise arbitrary, complex numbers. From the matrix H, of order n,

$$(1.1) \qquad H = \left\{ \frac{1}{a_i - b_j} \right\} \quad \text{with} \quad 1 \leq i, \quad j \leq n$$

and let $G = H^{-1} = \{c_{ij}\}$. (The indices i, j, k will range from 1 to n unless it is specified otherwise.) If, for some constant p,

$$(1.2) \qquad a_i - b_j = i + j - 1 + p \neq 0$$

then H is a segment of the well known generalized Hilbert matrix, and in this case formulas for c_{ij} have been given by Savage and Lukacs [4], Smith [6] and Collar [1]. For

$$(1.3) \qquad a_i - b_j = i - j + p$$

Linfoot and Shepherd [3] and Collar [2] give formulas for c_{ij} and in both cases Collar exhibits diagonal matrices D, K such that $G = DH^T K$. Collar [2] and Smith also evaluate the quantities $\sum_{i,j} c_{ij}$, $\sum_{j} c_{ij}$.

These authors make use, in most cases, of the formula for the determinant [5]

$$(1.4) \qquad \det H = \frac{\prod_{j>k}(a_j - a_k)(b_k - b_j)}{\prod_{j,k}(a_j - b_k)}$$

and/or require the evaluation of certain involved series.

The formulas of Collar and Smith are extended here to the general case (1.1). The method to be used does not depend on (1.4) but simply on Lagrange's interpolation formula. Indeed (1.4) comes out as a by-product of formula (3.3) given below.

2. Formula for the inverse.

Let

$$(2.1) \qquad A(x) = \prod_i (x - a_i), \qquad B(x) = \prod_i (x - b_i)$$

and denote the fundamental polynomials of the Lagrangian interpolation corresponding to the a_i and b_i, respectively, by

$$(2.2) \qquad A_i(x) = \frac{A(x)}{A'(a_i)(x - a_i)}, \quad B_i(x) = \frac{B(x)}{B'(b_i)(x - b_i)}$$

where prime denotes differentiation. We then have

Theorem 1. The elements of G are given by

$$(2.3) \qquad c_{ij} = (a_j - b_i)A_j(b_i)B_i(a_j)$$

and if H is symmetric

$$c_{ij} = (a_j - b_i)A_j(b_i)A_i(b_j). \tag{2.4}$$

Proof: For any polynomial $p(x)$ of degree $n-1$ we may write

$$p(x) = \sum_i p(a_i)\, A_i(x) \tag{2.5}$$

or

$$\frac{p(x)}{A(x)} = \sum_i \frac{c_i}{x - a_i} \tag{2.6}$$

where

$$c_i = \frac{p(a_i)}{A'(a_i)} \tag{2.7}$$

Now let $p(x) = -B_k(x)A(b_k) = p_k(x)$ and $c_{ki} = \dfrac{p_k(a_i)}{A'(a_i)}$.

Then from (2.6) we obtain

$$\frac{B_k(x)A(b_k)}{A(x)} = \sum_i \frac{c_{ki}}{a_i - x}. \tag{2.8}$$

If we set $x = b_j$ then, since $B_k(b_j) = \delta_{kj}$, the Kronecker delta, we get that

$$\delta_{kj} = \sum_i \frac{c_{ki}}{a_i - b_j}.$$

Thus c_{ki} gives the desired inverse element, that is,

$$c_{ki} = -\frac{B_k(a_i)A(b_k)}{A'(a_i)} = (a_i - b_k)B_k(a_i)A_i(b_k).$$

If H is symmetric: $a_i - b_j = a_j - b_i$, it follows that $B_i(a_j) = A_i(b_j)$ and the theorem is proved.

We now obtain a simple formula for the sum of the c_{ij}. This quantity arises in problems of aerodynamics [2] and its simplicity allows it also to be used as a check on some alleged inverse.

Corollary.

$$\sum_{i,j} c_{ij} = \sum_{k} (a_k - b_k) = s \tag{2.9}$$

Proof. If we apply (2.5) to $p(x) = B_j(x)$ and set $x = b_k$ we obtain that

$$\sum_{i} B_j(a_i)A_i(b_k) = \delta_{jk} \quad . \tag{2.10}$$

By symmetry this is also valid if A, a and B, b are interchanged. Thus we get that

$$s = \sum_{j} a_j \sum_{i} A_j(b_i)B_i(a_j) - \sum_{i} b_i \sum_{j} A_j(b_i)B_i(a_j)$$

$$= \sum_{j} a_j - \sum_{i} b_i \quad .$$

For the special case of the Hilbert segment (1.2), formula (2.9) gives Smith's [6] formula: $s = n(p+n)$ and for the matrix of (1.3) we get the formula of Collar [2]: $s = pn$.

3. Row and column sums.

Let the row and column sums of G be, respectively, (3.1) $$\sum_j c_{ij} = \alpha_i, \quad \sum_i c_{ij} = \beta_j$$

and define the diagonal matrices D_α, D_β by

$$D_\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n]$$
$$D_\beta = [\beta_1, \beta_2, \ldots, \beta_n]$$

Theorem 2. The matrix H then satisfies the following relations:

(3.2) $$\alpha_i = \frac{-A(b_i)}{B'(b_i)}, \quad \beta_j = \frac{B(a_j)}{A'(a_j)} .$$

(3.3) $$H^{-1} = D_\alpha H^T D_\beta .$$

Proof: Assuming (3.2) to be true we note that c_{ij} can be written in the form

(3.4) $$c_{ij} = \frac{1}{b_i - a_j} \frac{A(b_i)}{B'(b_i)} \frac{B(a_j)}{A'(a_j)} = \frac{\alpha_i \beta_j}{a_j - b_i}$$

which gives (3.3).

To prove the formula for α_i in (3.2) we need only show that

(3.5) $$\sum_j \frac{B(a_j)}{(a_j - b_i)\, A'(a_j)} = 1.$$

However for any function $f(x)$ we may write

(3.6) $$f[a_1, a_2, \ldots, a_n] \equiv \delta^{(n-1)} f(x) = \sum_j \frac{f(a_j)}{A'(a_j)}$$

where the left side of (3.6) is the $(n-1)^{th}$ divided difference of $f(x)$ with respect to the a_k (see Milne-Thomson [7] p. 9). If we apply (3.6) to the polynomial

$f(x) = B(x)/(x - b_i)$ of degree $n-1$ we have that $\delta^{(n-1)} f(x) \equiv$ constant. Since the coefficient of x^{n-1} in $f(x)$ is 1, (3.5) is proved.

An alternate proof of (3.5) may be obtained, without divided differences, by noting that the left hand side of (3.5) is the sum of the residues of the function $B(x)/A(x)(x - b_i)$ at the a_j. (This, in fact, follows readily from (2.6).) However for a sufficiently large circle C about the origin in the complex x- plane

$$\frac{1}{2\pi i} \int_C \frac{B(x)}{(x - b_i)A(x)}\, dx = 1$$

which proves (3.5).

The proof for the β_j is obtained in the same manner from (3.5) with the roles of A, a and B, b interchanged, and the theorem is proved. (The formula (3.5), incidentally, represents an extension of one of the formulas of Collar [2] for the sum of a series. Extensions of other formulas given in [2] can likewise be obtained from (3.6) by specializing $f(x)$.)

4. Remarks.

1.) We note that if H is symmetric then $\alpha_i = \beta_i$

and if we set $D = D_\alpha = D_\beta$ then

$$(4.1) \qquad G = D\,H\,D$$

2.) From (3.2) and (3.3) one immediately obtains the formula for the determinant of H up to a sign. That is

$$(4.2) \qquad (\det H)^2 = \prod_i \frac{1}{\alpha_i \beta_i}$$

and for H symmetric we get that

$$(4.3) \qquad \det H = (-1)^{\frac{n(n-1)}{2}} \prod_i \frac{A'(a_i)}{B(a_i)} .$$

The signs may readily be determined, by induction, by using (2.2) and the formula for the $(n+1,n+1)$ element of G_{n+1}:

$$c_{n+1,n+1} \cdot \det H_{n+1} = \det H_n$$

where $H_n = H$ and $H_{n+1} = \left\{\frac{1}{a_i - b_j}\right\}$ with $1 \leq i,j \leq n+1$.

3.) Formula (2.4) may also be applied to the problem of obtaining a least-squares fit of a function $f(x)$ on, say, $(0,1)$ by a function of the form $\sum_i \gamma_i x^{a_i}$. (All variables are here assumed to be real.) The normal equations for this problem yield a matrix of the form (1.1) with $b_i = -a_i - 1$, and the γ_i may be obtained by applying G to the moments of $f(x)$ with respect to the x^{a_i}.

For fitting a function of two variables the problem to determine γ_{ij} such that

$$\int_0^1 \int_0^1 [f(x,y) - \sum_{i,j} \gamma_{ij} \, x^{a_i} \, y^{a_i'}]^2 \, dx \, dy = \min$$

is solved by

$$K = D \, H \, D \, F \, D' \, H' D'$$

where $H = \left\{ \frac{1}{a_i + a_j + 1} \right\}$, $H' = \left\{ \frac{1}{a_i' + a_j' + 1} \right\}$, $K = \{ \gamma_{ij} \}$.

$F = \left\{ \int_0^1 \int_0^1 f(x,y) \, x^{a_i} \, y^{a_j'} \, dx \, dy \right\}$, and D, D' are the diagonal matrices corresponding to H, H', respectively, defined above.

Although the elements c_{ij} get quite large in the case of the Hilbert matrix, it may happen that for suitable choices of the a_i this may not be the case.

An explicit solution for γ_i can also be obtained from (2.3) for the equations

$$\int_0^1 (f(x) - \sum_i \gamma_i \, x^{a_i}) \, x^{a_k'} \, dx = 0, \quad k = 1, 2, \ldots, n$$

given the moments of $f(x)$ with respect to the $x^{a_i'}$.

4.) If H is real and symmetric and if $a_i > b_i$, $i = 1, 2, \ldots, n$ then it follows from (1.4) that all the principle minor determinants are positive and H is positive definite. In this case $a_i > b_j$ for all i

and j; that is, all the elements of H are positive, since if $a_i < a_j$, then $0 < a_i - b_i < a_j - b_i = a_i - b_j$. Thus $B(a_i) > 0$ and, since $A(x)$ has n simple zeros at the a_i, $A'(a_i)$ alternate in sign. From (2.4) it then follows that if $a_i < a_j$ for $i < j$ then $(-1)^{i+j} c_{ij} > 0$ so that then G has the same checkerboard distribution of signs as the inverse of the Hilbert segment. From (3.2) it follows that, in this case, the α_i also alternate in sign and that $(-1)^{j+n} \alpha_j > 0$.

Let λ and μ denote the smallest and largest eigenvalue of H respectively, and let $M_i = \min_{k, k \neq i} \left(\frac{a_k - b_i}{|a_k - a_i|} \right)$. Then it follows readily that $|A_i(b_i)| \geq M_i^{n-1} > 1$ and that $\lambda \leq \min_i \frac{1}{c_{ii}} \leq \min_i \left(\frac{M_i^{2-2n}}{a_i - b_i} \right)$, and if the a_i increase with i then $\min_i \left(\sum_j (-1)^{i+j} c_{ij} \right)^{-1} \leq \lambda$. We get for the P-condition [8] of H,

$$\frac{\mu}{\lambda} \geq \max_i \left(\frac{1}{a_i - b_i} \right) \cdot \max_i c_{ii} \geq \max_i [A_i(b_i)]^2 \geq \max_i M_i^{2n-2}$$

so that the P-condition of H may get very large. This number has been estimated for the Hilbert segment (1.2) with $p = 0$ by Todd [8].

5.) From the foregoing we may also get a simple proof of a lemma of Ky Fan and Gordon Pall [9] which, stated in a slightly more general form, says the following:

Let $a_1, a_2, \dots, a_{n+1}, b_1, \dots, b_n, t$ *be colinear complex numbers such that for a fixed* θ *and* t *and real numbers* ρ_j, τ_j

$$a_j = t + \rho_j e^{i\theta} \qquad 1 \le j \le n+1$$

$$b_j = t + \tau_j e^{i\theta} \qquad 1 \le j \le n$$

where

$$(4.4) \qquad \rho_1 > \tau_1 > \rho_2 > \tau_2 > \dots > \tau_n > \rho_{n+1} .$$

Then there exists a unique solution vector x *to the system*

$$(4.5) \qquad \sum_{k=1}^{n} \frac{x_n}{a_j - b_k} + x_{n+1} = a_j \; , \quad 1 \le j \le n+1$$

such that

$$x_j = \gamma_j e^{2i\theta}, \; \gamma_j > 0, \; 1 \le j \le n.$$

To prove this we let u, $\bar{x}$, $\bar{a}$, r be column vectors of order n such that

$$u_j = 1, \quad \bar{x}_j = x_j, \quad \bar{a}_j = a_j, \quad r_j = \frac{1}{a_{n+1} - b_j} \qquad 1 \le j \le n,$$

and write (4.5) in the form

$$(4.6) \qquad \begin{cases} H\bar{x} + u\, x_{n+1} = \bar{a} \\ r^T\bar{x} + x_{n+1} = a_{n+1} \end{cases}$$

where H is given by (1.1). By eliminating x_{n+1} we obtain

$$(4.7) \qquad (H - ur^T)\,\bar{x} = \bar{a} - ua_{n+1} \quad .$$

A single computation shows that

$$H - ur^T = L_1 H L_2^{-1}$$

where L_1 and L_2 are diagonal matrices given by

$$L_1 = [a_{n+1} - a_1, \ldots, a_{n+1} - a_n]$$
$$L_2 = [a_{n+1} - b_1, \ldots, a_{n+1} - b_n]$$

Thus it follows from (4.7) that

$$\bar{x} = L_2 H^{-1} L_1^{-1} (\bar{a} - u\,a_{n+1}) = -L_2 H^{-1} u = -L_2 \alpha$$

where α is the column vector with components α_j given by (3.2).

If we let $R(x) = \prod_{j=1}^{n} (x - \rho_j)$, $T(x) = \prod_{j=1}^{n} (x - \tau_j)$ and let $R_j(x)$, $T_j(x)$ be the corresponding Lagrangian coefficients then

$$\alpha_j = -\frac{A(b_j)}{B'(b_j)} = -\frac{R(\tau_j)}{T'(\tau_j)}\, e^{i\theta} \quad .$$

This implies that for $1 \leq j \leq n$

$$x_j = (\tau_j - \rho_{n+1}) \left(\frac{-R(\tau_j)}{T'(\tau_j)}\right) e^{2i\theta} = \gamma_j e^{2i\theta} \quad .$$

From (4.4) it follows readily that for $1 \leq j \leq n$

$$R(\tau_j) < 0 < T'(\tau_j) \quad \text{so that} \quad \gamma_j > 0 \quad .$$

In the particular case where a_j and b_j are real and t and θ are chosen to be zero we get the result as stated in [9] that there exists a unique x such that $x_j > o$ for $1 \leq j \leq n$.

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